Products of CW complexes

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Bringing set theory and algebraic topology together

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For $n \in \mathbb{N}$, let

- D^n denote the closed ball of radius 1 about the origin in $\mathbb{R}^n$ (the *n-disc*),
- $\mathring{D}^n$ its interior (the open ball of radius 1 about the origin), and
- S^{n-1} its boundary (the $n-1$-*sphere*).

CW complexes

Definition

A Hausdorff space X is a *CW complex* if there exists a set of continuous functions $\varphi^n_\alpha : D^n \to X$ (*characteristic maps*), for α in an arbitrary index set and $n \in \mathbb{N}$ a function of α, such that:

1. $\varphi^n_\alpha \restriction \mathring{D}^n$ is a homeomorphism to its image, and X is the disjoint union as α varies of these homeomorphic images $\varphi^n_\alpha[\mathring{D}^n]$ ("cells").

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3. *Weak topology:* A set is closed if and only if its intersection with each closed cell $\varphi_\alpha^n[D^n]$ is closed.

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We often denote $\varphi_\alpha^n[\mathring{D}^n]$ by e_α^n.

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Proof

Identify each edge with the unit interval, with x_0 at 0. Then for every $f : \mathbb{N} \to \mathbb{N}$, consider the open neighbourhood $U(x_0; f)$ of x_0 whose intersection with $e^1_{X,n}$ is the interval $[\,0, 1/(f(n) + 1)\,)$.

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These form a neighbourhood base, but for any countably many f_i, there is a g that eventually dominates each of them, so $U(x_0; g)$ does not contain any of the $U(x_0; f_i)$. □

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Convention

In this talk, $X \times Y$ is always taken to have the product topology, so "$X \times Y$ is a CW complex" means "the product topology on $X \times Y$ is the same as the weak topology".

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Then $\left(\frac{1}{g(k)+1}, \frac{1}{g(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

More preliminaries: subcomplexes

A *subcomplex* A of a CW complex X is what you would expect.

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A *subcomplex* A of a CW complex X is a subspace which is a union of cells of X, such that if $e^n_\alpha \subseteq A$ then its closure $\bar{e^n_\alpha} = \varphi^n_\alpha[D^n]$ is contained in A.

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Definition

Let κ be a cardinal. We say that a CW complex X is *locally less than κ* if for all x in X there is a subcomplex A of X with fewer than κ many cells such that x is in the interior of A. We write *locally finite* for locally less than $\aleph_0$, and *locally countable* for locally less than $\aleph_1$.

Proposition

If κ is a regular uncountable cardinal, then a CW complex W is locally less than κ if and only if every connected component of W has fewer than κ many cells.

Proof sketch.

$\Leftarrow$ is trivial. For $\Rightarrow$, given any point w, recursively fill out to get an open (hence clopen) subcomplex containing w with fewer than κ many cells, using the fact that the cells are compact to control the number of cells along the way. □

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If X and Y are both (locally) countable, then $X \times Y$ is a CW complex.

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Theorem (J. Milnor, 1956)

If X and Y are both (locally) countable, then $X \times Y$ is a CW complex.

Theorem (Y. Tanaka, 1982)

If neither X nor Y is locally countable, then $X \times Y$ is not a CW complex.

What was known, beyond ZFC

Theorem (Liu Y.-M., 1978)

Assuming CH, $X \times Y$ is a CW complex if and only if either

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Theorem (Y. Tanaka, 1982)

Assuming $\mathfrak{b} = \aleph_1$, $X \times Y$ is a CW complex if and only if either

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Can we do better?

Question

Can we show, without assuming any extra set-theoretic axioms, that the product $X \times Y$ of CW complexes X and Y is a CW complex if and only if either

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Answer (follows from Tanaka's work)

No.

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Answer (B.-T.)

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Recall:

For $f, g \in \mathbb{N}^\mathbb{N}$, write $f \leq^* g$ if for all but finitely many $n \in \mathbb{N}$, $f(n) \leq g(n)$.
I'll write $f \leq g$ to mean that for all n, $f(n) \leq g(n)$.

The bounding number $\mathfrak{b}$ is the least cardinality of a set of functions that is unbounded with respect to $\leq^*$, i.e. such that no one g is $\geq^*$ them all, i.e.,

$$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^\mathbb{N} \wedge \forall g \in \mathbb{N}^\mathbb{N} \exists f \in \mathcal{F} \neg(f \leq^* g)\}.$$

$\aleph_1 \leq \mathfrak{b} \leq 2^{\aleph_0}$, and each inequality can be strict.

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where we have identified each edge with the unit interval, with 0 at the centre vertex.

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Let $g \colon \mathbb{N} \to \mathbb{N}^+$ be an increasing function such that $[0, 1/g(n)) \subset e^1_{X,n} \cap U$ for every $n \in \mathbb{N}$. Take $f \in \mathcal{F}$ such that $f \not\leq^* g$.

Consider the edge $e^1_{Y,f}$ of Y:

Let $k \in \mathbb{N}$ be such that $\frac{1}{f(k)+1} \in e^1_{Y,f} \cap V$ and $f(k) > g(k)$.

Then $\left(\frac{1}{f(k)+1}, \frac{1}{f(k)+1} \right) \in U \times V \cap H$. So in the product topology, $(x_0, y_0) \in \bar{H}$.

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Yes!

A complete characterisation

Theorem (B.-T.)

Let X and Y be CW complexes. Then $X \times Y$ is a CW complex if and only if one of the following holds:

1. *X or Y is locally finite.*
2. *One of X and Y is locally countable, and the other is locally less than $\mathfrak{b}$.*

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So it remains to show that if X and Y are CW complexes such that X is locally countable and Y is locally less than $\mathfrak{b}$, then $X \times Y$ is a CW complex.

By the Proposition earlier, we may assume that X has countably many cells and Y has fewer than $\mathfrak{b}$ many cells.

Topologies

Any compact subset of a CW complex X is contained in finitely many cells, and each closed cell $\bar{e}^n_\alpha$ is compact. So

$$X \text{ has the weak topology } \Leftrightarrow \text{ the topology is } \textit{compactly generated}$$

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Definition

A topological space Z is *sequential* if for every subset C of Z, C is closed if and only if C contains the limit of every convergent (countable) sequence from C — C is *sequentially closed*.

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Any sequential space is compactly generated. Since D^n is sequential for every n, we have that CW complexes are sequential.

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So suppose

- $H \subset X \times Y$ is sequentially closed, and
- $(x_0, y_0) \in X \times Y \setminus H$.

We want to construct open neighbourhoods U of x_0 in X and V of y_0 in Y such that $(U \times V) \cap H = \emptyset$.

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- If $x \in e^n_\alpha \subset X$, start with the image under φ^n_α of an open ball in $\mathring{D}^n$.

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- Once $U \cap X^k$ is defined, for each $(k+1)$-cell e^{k+1}_β whose boundary intersects $U \cap X^k$, take a *collar neighbourhood* of $(\varphi^{k+1}_\beta)^{-1}(U \cap X^k) \subseteq S^k = \partial D^{k+1}$.

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Wrinkle in proof.

Use compactness of closed cells. □

Constructing neighbourhoods avoiding H

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Lemma 1 (Adding one cell to finite subcomplexes)

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Suppose

- W and Z are CW complexes,
- W' is a finite subcomplex of W,
- Z' is a finite subcomplex of Z,
- $U \subseteq W'$ is open in W',
- $V \subseteq Z'$ is open in Z', and
- H is a sequentially closed subset of $W \times Z$ such that the closure of $U \times V$ is disjoint from H.

Let e be a cell of Z whose boundary is contained in Z'. Then there is a $p \in \mathbb{N}$ such that, if $V^{e,p}$ is V extended by the width $1/(p+1)$ collar in e, then $U \times V^{e,p}$ has closure disjoint from H.

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Proof sketch.

Use compactness, normality and sequentiality of $W' \times (Z' \cup e)$. □

Back to the proof of the Theorem

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We shall construct functions $f\colon \mathbb{N} \to \mathbb{N}$ and $g\colon J \to \mathbb{N}$, where J is the index set for cells of Y, such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from H.

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Basic idea

Simultaneous induction on cell number on the X side (after enumerating the cells of X in a reasonable order) and dimension on the Y side.

For each new cell e_α that you consider on the Y side, you get a function $f_\alpha : \mathbb{N} \to \mathbb{N}$ defining an open set on the X side avoiding H. Since there are fewer than $\mathfrak{b}$ many α, they can be eventually dominated by a single function f, with respect to which the e_α part of the neighbourhood can be chosen.

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This doesn't work ($f_\alpha \leq^* f$ isn't good enough).

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Solution

Hechler conditions!

Making it work

The construction is actually by recursion on dimension on the Y side, and simultaneously, constructing f as the limit of a descending sequence of Hechler conditions, that is:

- finite initial segments of f, and
- promises to dominate some function F thereafter.

Making it work

Lemma 2 (Adding a Y-side cell, fitting X-side promises)

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Let

- Y' be a finite subcomplex of Y containing y_0,
- $F\colon \mathbb{N} \to \mathbb{N}$ be a function,
- $i \in \mathbb{N}$, and
- s be a function from the indices of Y' to $\mathbb{N}$ such that $U(x_0; F) \times U(y_0; s) \subseteq X \times Y'$ has closure disjoint from H,
- $Y'' = Y' \cup e_\alpha$ for some cell e_α of Y not in Y'.

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Let

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- $Y'' = Y' \cup e_\alpha$ for some cell e_α of Y not in Y'.

Then there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that

1. $f(n) \geq F(n)$ for all n in $\mathbb{N}$, and $f(n) = F(N)$ for all $n < i$,
2. for every $f'\colon \mathbb{N} \to \mathbb{N}$ such that $f' \geq^* f$ and $f' \geq F$, there is a $q \in \mathbb{N}$ such that $U(x_0; f') \times U(y_0; s \cup \{(\alpha, q)\})$ has closure disjoint from H.

Proof of Lemma 2

For every finite tuple r of length n such that $r \geq F \restriction n$, $U(x_0; r) \subset U(x_0; F)$, so $U(x_0; r) \times U(y_0; s)$ certainly has closure disjoint from H.

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By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from H.

Proof of Lemma 2

For every finite tuple r of length n such that $r \geq F \upharpoonright n$, $U(x_0; r) \subset U(x_0; F)$, so $U(x_0; r) \times U(y_0; s)$ certainly has closure disjoint from H.

By Lemma 1, we can then take $q_r \in \mathbb{N}$ such that $U(x_0; r) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from H.

Then by Lemma 1 again, there is $p \in \mathbb{N}$ sucht that $U(x_0; r \cup \{(n, p)\}) \times U(y_0; s \cup \{(\alpha, q_r)\})$ has closure disjoint from H.

Now, assuming by induction we have defined $f \restriction n$ ($n \geq i$), there are only finitely many r with $F \restriction n \leq r \leq f \restriction n$; follow this procedure for all of them, and take the maximum of the resulting values p to be $f(n)$.

Then for any $f' \geq F$ with $f' \geq^* f$, $f' \geq r \cup (f \restriction [n, \infty))$ for some $n \geq i$ and some r of length n as above, so

$$U(x_0; f' \restriction n+1) \times U(y_0; s \cup \{(\alpha, q_r)\}) \text{ has closure disjoint from } H,$$

and in fact

$$U(x_0; f') \times U(y_0; s \cup \{(\alpha, q_r)\}) \text{ has closure disjoint from } H.$$

□Lemma 2

Finishing the proof of the Theorem

With Lemma 2 in hand, the argument now follows as outlined before:

Proceed by induction of dimension on the Y side. Assume we have defined $f_k \colon \mathbb{N} \to \mathbb{N}$ and $g \upharpoonright Y^k$. For each $(k+1)$-dimensional cell e_α on the Y side, use Lemma 2 with f_k as F, k as i, the minimal (finite) subcomplex of Y containing e_α as Y'', and $g \upharpoonright (Y'' \smallsetminus e_\alpha)$ as s to get $f_{al,k+1}$. There are fewer than $\mathfrak{b}$ many such $f_{\alpha,k+1}$, so take f_{k+1} eventually dominating all of them. Then take q as given by Lemma 2 (with f_{k+1} as f') as $g(\alpha)$.

Finally, take f to be the (componentwise) limit of the f_{k+1}; these f and g are such that $U(x_0; f) \times U(y_0; g)$ has closure disjoint from H. $\square$